PMATH 352
Complex Analysis

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## 1 Calculus in the Plane

### 1.1 Review

Recall that the plane is the set

$$
\begin{equation*}
\mathbb{R}^{2}=\{(x, y) \mid x, y \in \mathbb{R}\} \tag{1}
\end{equation*}
$$

which is a 2-dimensional vector space (over $\mathbb{R}$ ) under standard component wise operations.

Notation 1.1. For points in $\mathbb{R}$ we will often use $x, y$. For vectors in $\mathbb{R}^{2}$ we use $\mathbf{x}=\left(x_{1}, x_{2}\right)$ and $\mathbf{y}=\left(y_{1}, y_{2}\right)$.

We equip it with the dot product (Euclidean inner product) by denoting

$$
\begin{equation*}
\mathbf{x} \cdot \mathbf{y}=x_{1} y_{1}+x_{2} y_{2} \tag{2}
\end{equation*}
$$

If $\mathbf{x} \in \mathbb{R}^{2}$, then $|\mathbf{x}|^{2}:=\mathbf{x} \cdot \mathbf{x}=\mathbf{x}^{T} \mathbf{x}=x_{1}^{2}+x_{2}^{2} \geq 0$. The distance from 0 to $\mathbf{x}$ is $|\mathbf{x}|=\sqrt{x_{1}^{2}+x_{2}^{2}}$.

Recall also the Cauchy-Schwarz and triangle inequalities:

$$
\begin{gather*}
|\mathbf{x} \cdot \mathbf{y}| \leq \mathbf{x} \cdot \mathbf{y}  \tag{3}\\
||\mathbf{x}|-|\mathbf{y}|| \leq|\mathbf{x}+\mathbf{y}| \leq|\mathbf{x}|+|\mathbf{y}| \tag{4}
\end{gather*}
$$

Notation 1.2. Let $\mathbf{x} \in \mathbb{R}^{2}$ and $r>0$, we define the open disk

$$
\begin{equation*}
D(\mathbf{x}, r):=\left\{\mathbf{y} \in \mathbb{R}^{2}| | \mathbf{x}-\mathbf{y} \mid<r\right\} \tag{5}
\end{equation*}
$$

Analagously we denote the closed disk by

$$
\begin{equation*}
\overline{D(\mathbf{x}, r)}:=\left\{\mathbf{y} \in \mathbb{R}^{2}| | \mathbf{x}-\mathbf{y} \mid \leq r\right\} \tag{6}
\end{equation*}
$$

Definition 1.3. Let $\Omega \subseteq \mathbb{R}^{2}$ with $\mathbf{x} \in \Omega$. We say that $\mathbf{x}$ is an interior point of $\Omega$ if $\exists \varepsilon>0$ such that $D(\mathbf{x}, \varepsilon) \subseteq \Omega$

Definition 1.4. $\Omega \subseteq \mathbb{R}^{2}$ is an open set if every $\mathrm{x} \in \Omega$ is an interior point of $\Omega$.

## Remark 1.5.

- $\varnothing$ is trivially open
- $\mathbb{R}^{2}$ is open
- open disks are open sets
- closed disks are not open sets

Example 1.6. Let $\Omega=D(\mathbf{x}, \varepsilon) \backslash\{\mathbf{x}\}$, then $\Omega$ is open (called the punctured disk).

## Recall 1.7.

(i) If $\Omega_{1}, \Omega_{2}$ are open then $\Omega_{1} \cup \Omega_{2}$ is open.
(ii) If $\Omega_{i}$ is open for all $i \in \mathcal{I}$ then $\bigcup_{i \in \mathcal{I}} \Omega_{i}$ is open.

Definition 1.8. Let $E \subseteq \mathbb{R}^{2}$, we say that $E$ is disconnected if there exist open sets $\Omega_{1}, \Omega_{2} \subseteq \mathbb{R}^{2}$ such that

$$
\begin{gather*}
E \cap \Omega_{1}, E \cap \Omega_{2} \neq \varnothing  \tag{7}\\
E \cap \Omega_{1} \cap \Omega_{2}=\varnothing  \tag{8}\\
\left(E \cap \Omega_{1}\right) \cup\left(E \cap \Omega_{2}\right)=E \tag{9}
\end{gather*}
$$

We say the $E$ is connected if it is not disconnected.
Recall 1.9. Let $F: E \subseteq \mathbb{R}^{2} \rightarrow \mathbb{R}$ be continuous on $E$. If $E$ is connected, then $F(E)=\{F(\mathbf{x}) \mid \mathbf{x} \in E\}$ is an interval (continuous map of connected sets is connected).

Corollary 1.10. (IVT) Let $F: E \rightarrow \mathbb{R}$ be continuous, let $\mathbf{x}_{1}, \mathbf{x}_{2} \in F(E)$, and let $t_{1}=F\left(\mathbf{x}_{1}\right), t_{2}=F\left(\mathbf{x}_{2}\right)$. Then for all $t$ between $t_{1}$ and $t_{2}$ there exists a $\mathbf{x} \in E$ such that $F(\mathbf{x})=t$.

Definition 1.11. A domain $\Omega$ in $\mathbb{R}^{2}$ is a non-empty open connected set.
Theorem 1.12. Let $\Omega \subseteq \mathbb{R}^{2}$ be open, then the following are equivalent:

- $\Omega$ is connected
- Any pair of points in $\Omega$ can be connected by a finite number of straight line segments in $\Omega$

Proof. Let $\mathbf{x} \in \Omega$ and let

$$
S=\{\mathbf{y} \in \Omega \mid \text { exists a piecewise linear path from } \mathbf{x} \text { to } \mathbf{y} \text { in } \Omega\}
$$

Claim. $S$ is open
Proof. For $\mathbf{z} \in S$ we have, since $\mathbf{z} \in \Omega$ and $\Omega$ is open, an $\varepsilon>0$ such that $D(\mathbf{z}, \varepsilon) \subseteq \Omega$ and each point in $D(\mathbf{z}, \varepsilon)$ is reachable from $\mathbf{z}$ with one additional line segment, hence $D(\mathbf{z}, \varepsilon) \subseteq S$.

Claim. $\Omega \backslash S$ is open
Proof. Let $\mathbf{z} \in \Omega \backslash S$. Since $\mathbf{z} \in \Omega$ and $\Omega$ is open there is an $\varepsilon>0$ such that $D(\mathbf{z}, \varepsilon) \subseteq \Omega$. If $D(\mathbf{z}, \varepsilon) \cap S \neq \varnothing$ there would be a polygonal path from $\mathbf{x}$ to a point in $D(\mathbf{z}, \varepsilon)$ and hence to $\mathbf{z}$. Hence $D(\mathbf{z}, \varepsilon) \cap S=\varnothing$, so $D(\mathbf{z}, \varepsilon) \subseteq \Omega \backslash S$ and $\Omega \backslash S$ is open.

Now let $E=\Omega, \Omega_{1}=S, \Omega_{2}=\Omega \backslash S$. Then we have

$$
\begin{gather*}
E \cap \Omega_{1}=S \neq 0  \tag{10}\\
E \cap \Omega_{2}=\Omega \backslash S  \tag{11}\\
E \cap \Omega_{1} \cap \Omega_{2}=\varnothing  \tag{12}\\
E=\Omega_{1} \cup \Omega_{2} \tag{13}
\end{gather*}
$$

but $\Omega$ was assumed connected, so $\Omega \backslash S$ must be empty.

Now suppose $\Omega$ is open and piecewise linear path connected. Assume for contradiction that $\Omega$ is not connected and let $\Omega_{1}, \Omega_{2}$ be a disconnection, that is open sets such that:

$$
\begin{gather*}
\Omega_{1}, \Omega_{2} \neq \varnothing  \tag{14}\\
\Omega_{1} \cap \Omega_{2}=\varnothing  \tag{15}\\
\Omega_{1} \cup \Omega_{2}=\Omega \tag{16}
\end{gather*}
$$

Let $\mathbf{x}_{1} \in \Omega_{1}, \mathbf{x}_{2} \in \Omega_{2}$. By assumption there exists a piecewise linear path from $\mathbf{x}_{1}$ to $\mathbf{x}_{2}$ in $\Omega$, hence there is a continuous map $\alpha:[0,1] \rightarrow \mathbb{R}^{2}$ such that $\alpha(0)=\mathbf{x}_{1}$ and $\alpha(1)=\mathbf{x}_{2}$. Let $E=\alpha([0,1])$, then $E$ is connected as it is the continuous map of a connected set. Then we have that

$$
\begin{gather*}
E \cap \Omega_{1}, E \cap \Omega_{2} \neq \varnothing  \tag{17}\\
E \cap \Omega_{1} \cap \Omega_{2}=\varnothing  \tag{18}\\
\left(E \cap \Omega_{1}\right) \cup\left(E \cap \Omega_{2}\right)=E \cap\left(\Omega_{1} \cup \Omega_{2}\right)=E \tag{19}
\end{gather*}
$$

so $\Omega_{1}, \Omega_{2}$ give a disconnection of $E$, a connected set. We conclude that $\Omega$ is open.

We now give some examples of domains:

## Example 1.13.

- Any open convex set is a domain, hence $D(\mathbf{x}, r)$ is always a domain.
- A punctured disk $D(\mathbf{x}, r) \backslash\{\mathbf{x}\}$ is a domain.
- An annulus $\left\{\mathbf{y} \in \mathbb{R}^{2}\left|R_{1}<|\mathbf{x}-\mathbf{y}|<R_{2}\right\}\right.$ centered at $\mathbf{x} \in \mathbb{R}^{2}$ is a domain.

Definition 1.14. A set $E \subseteq \mathbb{R}^{2}$ is bounded if there is an $r \in \mathbb{R}$ such that $E \subseteq D(\mathbb{O}, r)$.

Notation 1.15. For a set $E \subseteq \mathbb{R}^{2}$ we write $E^{C}:=\mathbb{R}^{2} \backslash E$ to denote the complement.

Definition 1.16. A set $E \subseteq \mathbb{R}^{2}$ is closed if its complement $E^{C}$ is open.
This immediately implies from (1.7) that intersections of closed sets are closed and finite unions of closed sets are closed.

Definition 1.17. A set $E \subseteq \mathbb{R}^{2}$ is compact if it is closed and bounded
Recall 1.18. Continuous images of compact sets are compact.
Corollary 1.19. (EVT) Let $f: E \subseteq \mathbb{R}^{2} \rightarrow \mathbb{R}$ bs continuous on $E$, with $E$ compact. Then $f$ attains a global max and min on $E$.

Definition 1.20. Let $E \subseteq \mathbb{R}^{2}$ and let $\mathbf{x} \in E$. We say that $\mathbf{x}$ is a boundary point of $E$ if $\varepsilon>0$ we have

$$
\begin{equation*}
D(\mathbf{x}, \varepsilon) \cap E, D(\mathbf{x}, \varepsilon) \cap E^{C} \neq \varnothing \tag{20}
\end{equation*}
$$

## Example 1.21.

- Let $E=D(\mathbf{x}, r)$, then each $\mathbf{y} \in\left\{\mathbf{z} \in \mathbb{R}^{2}| | \mathbf{x}-\mathbf{z} \mid=r\right\}$ is a boundary point of $E$.
- Let $E=\overline{D(\mathbf{x}, r)}$, then the boundary points are the same as the prior example.

Definition 1.22. For a set $E \subseteq \mathbb{R}^{2}$ we write $\partial E$ to denote the set of all boundary points of $E$.

## Remark 1.23.

- A set $E$ is open iff $E \cap \partial E=\varnothing$ and closed iff $E \cap \partial E=E$
- For a set $E$, we have $\partial E=\partial E^{C}$

Example 1.24. Let $\Omega=\left\{(x, y) \in \mathbb{R}^{2} \mid y>0\right\}$ (upper half plane). Then $\Omega$ is a domain and $\partial \Omega$ is the $x$-axis.

### 1.2 Curves in $\mathbb{R}^{2}$

Definition 1.25. A smooth curve in $\mathbb{R}^{2}$ is a map $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ such that $\alpha(t)=\left(\alpha_{1}(t), \alpha_{2}(t)\right)$ where:

$$
\begin{gather*}
\alpha \in \mathcal{C}^{1}([a, b])  \tag{21}\\
\alpha^{\prime}(t)=\left(\alpha_{1}^{\prime}(t), \alpha_{2}^{\prime}(t)\right) \neq \mathbb{0} \tag{22}
\end{gather*}
$$

Example 1.26. Let $R>0$, then $\alpha(t)=(R \cos (t), R \sin (t))$ for $t \in[0,2 \pi]$ is a smooth curve. Notice that $\left|\alpha^{\prime}(t)\right|=R>0$, and $\alpha$ is obviously smooth.

Definition 1.27. A piecewise smooth curve in $\mathbb{R}^{2}$ is a continuous map $\alpha$ : $[a, b] \rightarrow \mathbb{R}^{2}$ and a decomposition

$$
\begin{equation*}
a=t_{0}<t_{1}<\cdots<t_{n}=b \tag{23}
\end{equation*}
$$

such that $\left.\alpha\right|_{\left[t_{i-1}, t_{i}\right]}:\left[t_{i-1}, t_{i}\right] \rightarrow \mathbb{R}^{2}$ is a smooth curve.
Definition 1.28. A curve $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ is simple if both $\left.\alpha\right|_{(a, b]},\left.\alpha\right|_{[a, b)}$ are injective.

Definition 1.29. A curve $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ is closed if $\alpha(a)=\alpha(b)$.
Let $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ be a curve in $\mathbb{R}^{2}$, then
Definition 1.30. The length of $\alpha$ is

$$
\begin{equation*}
L(\alpha)=\int_{a}^{b}\left|\alpha^{\prime}(t)\right| \mathrm{d} t \tag{24}
\end{equation*}
$$

Example 1.31. Let $\alpha(t)=(R \cos (t), R \sin (t))$, then we have that $\alpha^{\prime}(t)=$ $(-R \sin (t), R \cos (t))$ and so $\left|\alpha^{\prime}(t)\right|=R$. We then get that

$$
L(\alpha)=\int_{0}^{2 \pi}\left|\alpha^{\prime}(t)\right| \mathrm{d} t=2 \pi R
$$

Definition 1.32. A reparametrization of a curve $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ is a smooth bijective map $h:[c, d] \rightarrow[a, b]$ such that $h$ is piecewise smooth and either

$$
\left\{\begin{array}{lll}
h^{\prime}(t)>0 & t \in[c, d] & \text { orientation preserving } \\
h^{\prime}(t)<0 & t \in[c, d] & \text { orientation reversing }
\end{array}\right.
$$

so that $\widetilde{\alpha}:[c, d] \rightarrow \mathbb{R}^{2}$ given by $\widetilde{\alpha}(s)=\alpha \circ h(s)$ is a piecewise smooth curve whose image is the same as that of $\alpha$ and that passes through each point in the image the same number of times as $\alpha$ and in the same direction (if $h$ preserves orientation) or the opposite direction (if $h$ reverses orientation).

Note that $\widetilde{\alpha}$ must be smooth by the chain rule. Also note that if $h$ is orientation preserving we have $\widetilde{\alpha}(c)=\alpha(a)$ and $\widetilde{\alpha}(d)=\alpha(b)$ and viceversa if $h$ is orientation reversing.
Proposition 1.33. Let $\widetilde{\alpha}=\alpha \circ h$ be a reparematerization of $\alpha$, then $L(\alpha)=$ $L(\widetilde{\alpha})$.

Proof. Notice first that

$$
\begin{equation*}
\left|\widetilde{\alpha}^{\prime}(s)\right|=\left|\alpha^{\prime}(h(s))\right|\left|h^{\prime}(s)\right| \tag{25}
\end{equation*}
$$

so we have:

$$
\begin{align*}
L(\widetilde{\alpha}) & =\int_{c}^{d}\left|\widetilde{\alpha}^{\prime}(s)\right| \mathrm{d} s  \tag{26}\\
& =\int_{c}^{d}\left|\alpha^{\prime}(h(s))\right|\left|h^{\prime}(s)\right| \mathrm{d} s  \tag{27}\\
& =\left\{\begin{array}{cc}
\int_{c}^{d}\left|\widetilde{\alpha}^{\prime}(s)\right| \frac{\mathrm{d} h}{\mathrm{~d} s} \mathrm{~d} s & h^{\prime}(s)>0 \\
\int_{c}^{d}\left|\widetilde{\alpha}^{\prime}(s)\right|-\frac{\mathrm{d} h}{\mathrm{~d} s} \mathrm{~d} s & h^{\prime}(s)<0
\end{array}\right.  \tag{28}\\
& =\int_{a}^{b}\left|\alpha^{\prime}(t)\right| \mathrm{d} t  \tag{29}\\
& =L(\alpha) \tag{30}
\end{align*}
$$

Theorem 1.34. (Reparameterization by arclength)
Let $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ be a curve, then there exists a unique orientation preserving reparameterization $h:[0, L(\alpha)] \rightarrow[a, b]$ such that $\widetilde{\alpha}:=\alpha \circ h$ has unit speed $\left(\left|\widetilde{\alpha}^{\prime}(s)\right|=1\right)$.

Proof. Assume $\alpha$ is smooth, we seek a bijection $h:[0, L] \rightarrow[a, b]$ such that $\widetilde{\alpha} \circ h(s)$ has unit speed.

Let $t=h(s) \Longrightarrow s=h^{-1}(t):=f(t)$. Note that $f(t)$ must be $\int_{a}^{t}\left|\alpha^{\prime}(u)\right| \mathrm{d} u$. We know that $f$ is differentiable with $f^{\prime}(t)=\left|\alpha^{\prime}(t)\right|$ and there exists an inverse $h$ such that $h \in \mathcal{C}^{1}([a, b])$. We also have that

$$
\begin{equation*}
f^{\prime}(h(s)) h^{\prime}(s)=1 \Longrightarrow h^{\prime}(s)=\frac{1}{f(h(s))}=\frac{1}{f^{\prime}(t)}=\frac{1}{\left|\alpha^{\prime}(t)\right|} \tag{31}
\end{equation*}
$$

Set $\widetilde{\alpha}(s):=\alpha(h(s))$, by the above $\widetilde{\alpha}^{\prime}(s)=1$. Suppose $\alpha$ is piecewise smooth, then we do the same thing on each interval $\left[t_{i-1}, t_{i}\right]$. Uniqueness follows by uniqueness of antiderivative.

Note 1.35. From now on, all curves are parameterized by arclength.
Definition 1.36. A Jordan curve is a curve $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ that is simple and closed

Theorem 1.37. (Jordan Curve Theorem)
Let $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ be a Jordan curve with image $\Gamma=\alpha([a, b])$. Then $\mathbb{R}^{2} \backslash \Gamma$ consists of 2 disjoint domains, one of which is bounded and one of which is unbounded. Each domain has $\Gamma$ as its boundary. If a point inside $\Gamma$ is connected to a point outside $\Gamma$ by a curve, then the curve must intersect $\Gamma$.

Proof. (ommited)
Definition 1.38. A Jordan domain is a bounded domain $\Omega$ such that its boundary is the union of finitely many images of Jordan curves. We choose to orient each of these Jordan curves so that, as we traverse the curve in the direction of its orientation, the Jordan domain lies on the left side.

### 1.3 Vector fields and line integrals

Let $\Omega$ be a Jordan domain with boundary $\partial \Omega$ and let $\Gamma$ be one of the Jordan curves in $\partial \Omega$. WLOG, let $\Gamma$ be parameterized by arclength. We then have that $\alpha(s)=\left(\alpha_{1}(s), \alpha_{2}(s)\right)$ for $s \in[0, L]$ where $L=L(\alpha)$. Let $T(s)=\alpha^{\prime}(s)=$ ( $\left.\alpha_{1}^{\prime}(s), \alpha_{2}^{\prime}(s)\right)$ be the unit tangent vector field to $\alpha$.

Definition 1.39. The outward normal vector field $N$ to $\alpha$ is defined to be $N(s)=\left(\alpha_{2}^{\prime}(s),-\alpha_{1}^{\prime}(s)\right)$. Note that we have $|N(s)|=|T(s)|=1$ so it makes sense to call this a unit vector field. Note also that $\langle N(s), T(s)\rangle=0$, so it is perpendicular to $T$, and moreover is obtained by rotating the vector field $T 90^{\circ}$ clockwise (so that $N(s)$ points outwards according to our convention for Jordan domains).

Example 1.40. Let $\alpha(s)=\left(R \cos \left(\frac{s}{R}\right), R \sin \left(\frac{s}{R}\right)\right)$, then

$$
\begin{gathered}
T(s)=\alpha^{\prime}(s)=\left(-\sin \left(\frac{s}{R}\right), \cos \left(\frac{s}{R}\right)\right) \\
N(s)=\left(\cos \left(\frac{s}{R}\right),-\sin \left(\frac{s}{R}\right)\right)
\end{gathered}
$$

Let $\Omega$ be a Jordan domain and let $\mathbf{z}_{0} \in \partial \Omega$ and let $N\left(\mathbf{z}_{0}\right)$ be the outward pointing unit normal vector. Let $W \supseteq \Omega \cup \partial \Omega$. Let $u \in \mathcal{C}^{1}(W)$ be a continuously differentiable function from $W$ to $\mathbb{R}$.

Definition 1.41. We write $\frac{\partial u}{\partial n}\left(\mathbf{z}_{0}\right):=D_{N\left(\mathbf{z}_{0}\right)} u=\nabla u\left(\mathbf{z}_{0}\right) \cdot N\left(\mathbf{z}_{0}\right)$ for the directional derivative of $u$ at $\mathbf{z}_{0}$ in the direction $N\left(\mathbf{z}_{0}\right)$.

Definition 1.42. (Laplacian)
Let $W$ be open and let $u \in \mathcal{C}^{2}(W)$. We define the Laplacian of $u$ to be

$$
\begin{equation*}
\Delta u:=u_{x x}+u_{y y} \in \mathcal{C}^{0}(W) \tag{32}
\end{equation*}
$$

Definition 1.43. Let $\Omega$ be open, a vector field $F$ on $\Omega$ is a map $F: \Omega \rightarrow \mathbb{R}^{2}$ given by $F(x, y)=(P(x, y), Q(x, y))$ where $P, Q: \Omega \rightarrow \mathbb{R}$. We say that $F$ is a $\mathcal{C}^{k}$ vector field on $\Omega$ if both $P, Q \in \mathcal{C}^{k}(\Omega)$. We always assume $F$ is atleast $\mathcal{C}^{0}$.

## Example 1.44.

- $F(x, y)=(-y, x)$
- $F(x, y)=\left(\frac{x}{\sqrt{x^{2}+y^{2}}}, \frac{y}{\sqrt{x^{2}+y^{2}}}\right)$

Let $\alpha:[a, b] \rightarrow \mathbb{R}^{2}$ be a curve and suppose that $\alpha([a, b]) \subseteq \Omega$. Let $F$ be a $\mathcal{C}^{0}$ vector field on $\Omega$. Suppose that $\alpha(t)=\left(\alpha_{1}(t), \alpha_{2}(t)\right)=(x(t), y(t))$

Definition 1.45. The line integral of the vector field $F$ along the curve $\alpha$ is defined to be

$$
\begin{align*}
\int_{\alpha} F \cdot \mathrm{~d} r & :=\int_{a}^{b} F(\alpha(t)) \cdot \alpha^{\prime}(t) \mathrm{d} t  \tag{33}\\
& =\int_{a}^{b} P(x(t), y(t)) \cdot x^{\prime}(t) \mathrm{d} t+\int_{a}^{b} Q(x(t), y(t)) \cdot y^{\prime}(t) \mathrm{d} t \tag{34}
\end{align*}
$$

Notation 1.46. Some authors write $\int_{\alpha} P \mathrm{~d} x+Q \mathrm{~d} y$ for the same line integral.
Example 1.47. Let $\alpha(t)=(R \cos (t), R \sin (t))$ and $F(x, y)=(-y, x)$. then we have

$$
\int_{\alpha} F \cdot \mathrm{~d} r=\int_{0}^{2 \pi} R^{2} \mathrm{~d} t=2 \pi R^{2}
$$

Proposition 1.48. The line integral is independent of reparameterization as long as the orientation is preserved.

Proof. Let $\widetilde{\alpha}(s)=\alpha(h(s))$ for $c \leq s \leq d$ be a reparameterization. We have $\widetilde{\alpha}^{\prime}(s)=\alpha^{\prime}(h(s)) h^{\prime}(s)$. Then we have

$$
\begin{align*}
\int_{\alpha} F \cdot \mathrm{~d} t & =\int_{a}^{b} F(\alpha(t)) \alpha^{\prime}(t) \mathrm{d} t  \tag{35}\\
& =\int_{c}^{d} F(\alpha(h(s))) \alpha^{\prime}(h(s)) h^{\prime}(s) \mathrm{d} s  \tag{36}\\
& =\int_{c}^{f} F(\widetilde{\alpha}(s)) \widetilde{\alpha}^{\prime}(s) \mathrm{d} s  \tag{37}\\
& =\int_{\widetilde{\alpha}^{\prime}} F \cdot \mathrm{~d} r \tag{38}
\end{align*}
$$

From the above proof, we can easily see that

$$
\begin{equation*}
\int_{\alpha} F \cdot \mathrm{~d} r=-\int_{-\alpha} F \cdot \mathrm{~d} r \tag{39}
\end{equation*}
$$

where $-\alpha$ is the reversal of the curve $\alpha$.

### 1.4 Green's Theorem and Green's Identities

Theorem 1.49. (Green's Theorem)
Let $\Omega$ be a $k$-connected Jordan domain and let $F$ be a $\mathcal{C}^{1}$ vector field on a domain $\Omega^{+} \supseteq \bar{\Omega}$. Then

$$
\begin{equation*}
\int_{\partial \Omega} F \cdot \mathrm{~d} r=\iint_{\Omega}\left(Q_{x}-P_{y}\right) \mathrm{d} A \tag{40}
\end{equation*}
$$

Proof. (ommitted)
Fix for now that $\Omega$ is a domain and $\Omega^{+}$is a domain containing $\bar{\Omega}$. Also let $u, v \in \mathcal{C}^{2}\left(\Omega^{+}\right)$be functions $u, v: \Omega^{+} \rightarrow \mathbb{R}$.

Definition 1.50. We define $\int_{\partial \Omega} \frac{\partial u}{\partial n} \mathrm{~d} s$ as follows. Let $F(x, y)=\left(-u_{y}, u_{x}\right)$ and note this is a $\mathcal{C}^{0}$ vector field. Let $\alpha:[0, L] \rightarrow \mathbb{R}^{2}$ be the arclength parameterization of $\partial \Omega$. Then we have

$$
\begin{align*}
\int_{\alpha} F \cdot \mathrm{~d} r & =\int_{0}^{L} F(\alpha(t)) \alpha^{\prime}(t) \mathrm{d} t  \tag{41}\\
& =\int_{0}^{L}\left(-u_{y} x^{\prime}+u_{x} y^{\prime}\right) \mathrm{d} t  \tag{42}\\
& =\int_{0}^{L}\left(u_{x}, u_{y}\right) \cdot\left(y^{\prime},-x^{\prime}\right) \mathrm{d} t  \tag{43}\\
& =\int_{0}^{L} \nabla u(\alpha(t)) \cdot N(t) \mathrm{d} t  \tag{44}\\
& =\int_{0}^{L} \frac{\partial u}{\partial n}(t) \mathrm{d} t \tag{45}
\end{align*}
$$

Theorem 1.51. (Green's First Identity)

$$
\begin{equation*}
\iint_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} A=\int_{\partial \Omega} u \frac{\partial v}{\partial n} \mathrm{~d} s-\iint_{\Omega} u \Delta v \mathrm{~d} A \tag{46}
\end{equation*}
$$

Proof. Let $F(x, y)=\left(-u v_{y}, u v_{x}\right)$, then we have

$$
\begin{align*}
\int_{\partial \Omega} u \frac{\partial v}{\partial n} \mathrm{~d} s & =\int_{\partial \Omega}\left(-u v_{y} \mathrm{~d} x+u v_{x} \mathrm{~d} y\right)  \tag{47}\\
& =\int_{\partial \Omega} F \cdot \mathrm{~d} r  \tag{48}\\
& =\iint_{\Omega}\left(Q_{x}-P_{y}\right) \mathrm{d} A  \tag{49}\\
& =\iint\left(u_{x} v_{x}+u v_{x x}-\left(-u_{y} v_{y}-u v_{y y}\right)\right) \mathrm{d} A  \tag{50}\\
& =\iint_{\Omega}[\nabla u \cdot \nabla v+u \Delta v] \mathrm{d} A \tag{51}
\end{align*}
$$

Theorem 1.52. (Green's Second Identity)

$$
\begin{equation*}
\int_{\partial \Omega}\left(v \frac{\partial u}{\partial n}-u \frac{\partial v}{\partial n}\right) \mathrm{d} s=\iint_{\Omega}(v \Delta u-u \Delta v) \mathrm{d} A \tag{52}
\end{equation*}
$$

Proof. Interchange $u$ and $v$ on LHS side of \#1 and subtract together.
Corollary 1.53. (Inside-Outside Theorem)

$$
\begin{equation*}
\int_{\partial \Omega} \frac{\partial v}{\partial n} \mathrm{~d} s=\iint_{\Omega} \Delta v \mathrm{~d} A \tag{53}
\end{equation*}
$$

Proof. Let $u=1$ above.
Lemma 1.54. Let $\mathbf{x}_{0} \in \mathbb{R}^{2}$ be fixed and let $\mathbf{x} \in \mathbb{R}^{2}$ be a variable point. Define $r(\mathbf{x}):=\left|\mathbf{x}-\mathbf{x}_{0}\right|$. Then $\log r(\mathbf{x})$ is harmonic on $\mathbb{R}^{2} \backslash\{\mathbf{x}\}$.

Proof. Compute the Laplacian
Theorem 1.55. (Green's Third Identity) Fix $\mathbf{x}_{0} \in \mathbb{R}^{2}$ and let $r(\mathbf{x})$ be as above (in terms of $\mathbf{x}_{0}$ ), then

$$
\begin{equation*}
u\left(\mathbf{x}_{0}\right)=\frac{1}{2 \pi} \iint_{\Omega} \log (r) \Delta u \mathrm{~d} A-\frac{1}{2 \pi} \int_{\partial \Omega}\left(\log (r) \frac{\partial u}{\partial n}-v \frac{\partial \log (r)}{\partial n}\right) \mathrm{d} s \tag{54}
\end{equation*}
$$

Proof. Let $\varepsilon>0$ such that $\overline{D\left(\mathbf{x}_{0}, \varepsilon\right)} \subseteq \Omega$. Apply Green's second identity to $\Omega \backslash \overline{D\left(\mathbf{x}_{0}, \Omega\right)}$ with $v=\log r$. We have that $\partial\left(\Omega \backslash \overline{D\left(\mathbf{x}_{0}, \varepsilon\right)}\right)=\partial \Omega \cup C\left(\mathbf{x}_{0}, \varepsilon\right)$ and on $C\left(\mathbf{x}_{0}, \varepsilon\right)$ we have that $v=\log \varepsilon$. We also have

$$
\begin{equation*}
\frac{\partial u}{\partial n}=-\frac{\partial v}{\partial r}=-\frac{1}{r} \tag{55}
\end{equation*}
$$

so we get that

$$
\begin{align*}
\int_{\partial \Omega}\left(\log r \frac{\partial u}{\partial n}-u \frac{\partial}{\partial n}(\log r)\right) \mathrm{d} s & +\int_{C\left(\mathbf{x}_{0}, \varepsilon\right)}\left((\log \varepsilon) \frac{\partial u}{\partial n}-\left(\frac{-1}{\varepsilon}\right) u\right) \mathrm{d} s  \tag{56}\\
& =\iint_{\Omega \backslash \overline{D\left(\mathbf{x}_{0}, \varepsilon\right)}}(\log r) \Delta u \mathrm{~d} A \tag{57}
\end{align*}
$$

Now, on $C\left(\mathbf{x}_{0}, \varepsilon\right)$ we have $s=\varepsilon \theta$, so $\mathrm{d} s=\varepsilon \mathrm{d} \theta$. So

$$
\begin{align*}
\int_{C\left(\mathbf{x}_{0}, \varepsilon\right)}\left[\log \varepsilon \frac{\partial u}{\partial n}+\frac{1}{\varepsilon} u\right] \mathrm{d} s & =\int_{0}^{2 \pi}\left[\log \varepsilon \frac{\partial u}{\partial n}+\frac{1}{\varepsilon} u\right] \varepsilon \mathrm{d} \theta  \tag{58}\\
& =\varepsilon \log \varepsilon \int_{0}^{2 \pi} \frac{\partial u}{\partial n} \mathrm{~d} \theta+\int_{0}^{2 \pi} u(\varepsilon, \theta) \mathrm{d} \theta  \tag{59}\\
& =\varepsilon \log \varepsilon K+2 \pi u\left(\varepsilon, \theta_{\varepsilon}\right)  \tag{60}\\
& \rightarrow 2 \pi u\left(\mathbf{x}_{0}\right) \text { as } \varepsilon \rightarrow 0 \tag{61}
\end{align*}
$$

and the result follows by rearrangement.

## 2 Harmonic Functions

