PMATH465 Smooth Manifolds

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Abstract

Smooth manifold theory is the first step towards differential geometry. We start with linear algebra, considering finite dimensional real vector spaces and linear maps between them (ie: $F : \mathbb{R}^n \cong V \to V \cong \mathbb{R}^m$). We then get to calculus where we still consider maps from $\mathbb{R}^n \to \mathbb{R}^m$, albeit with the linearity condition lifted - instead we restrict to maps with good smoothness (for some notion of smoothness) (ie: if $F : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^1$ near some $p \in \mathbb{R}^n$ then F is close to its linearization/differential/total derivative at p; that is, $(DF)_p : \mathbb{R}^n \to \mathbb{R}^m$ is linear and a good approximation of F near p) (to make things easier, we will often assume that maps are $\mathcal{C}^\infty$). Finally, for smooth manifold theory we allow the spaces (domain and codomain of F) to be "nonlinear" (ie: not vector spaces) but now we want our spaces to be locally well approximated by linear spaces (whereas in calculus we wanted our maps to be well approximated by linear maps near a point).

1 Overview

We will start by being vague for the sake of intuition and formalize these notions later. Roughly speaking, a **manifold** is a "space" that is well approximated by a linear vector space near each point. The prototypical picture for maps between manifolds is:

./manifoldmap.png

Near p, M is "almost" linear and near $F(p)$, N is "almost" linear. F is "almost" a linear map between a neighborhood (nbhd) of p in M and a nbhd of $F(p)$ in N.

Essentially, a manifold is a "space" on which we can do calculus[1] (this isn't quite true, integration will work and differentiating once will work[2]). Note that we can impose additional structure on a smooth manifold (still undefined) to be able to do differential geometry.

2 Topology and Topological Manifolds

We'll begin with general topological spaces[3]. In this section we will cover topological spaces, continuity, subspaces, product spaces, quotient spaces, the Hausdorff property, connectedness, compactness, etc.

2.1 Topological Spaces

Definition Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets on X, called **open sets** satisfying:

(i) $\varnothing, X \in \mathcal{T}$

(ii) If $U_1, \ldots, U_k \in \mathcal{T}$ then $\bigcap_{i=1}^{k} U_i \in \mathcal{T}$ (finite intersections of open sets are open)

(iii) If $U_\alpha \in \mathcal{T}$ for $\alpha \in A$ then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ (arbitrary unions of open sets are open)

The pair $(X, \mathcal{T})$ is called a **topological space**. Sometimes, when the topology is understood by the context, we simply write X.

Remark A given set X can have many different topologies.

Example The standard example which guides our intuition is $\mathbb{R}^n$ with the usual (metric) topology where a subset U is open iff $\forall_{p \in U} \exists_{\varepsilon > 0} B(p, \varepsilon) \subseteq U$.

Exercise Show that the metric topology on $\mathbb{R}^n$ is a topology in the sense of the above definition.

More generally, the following are also topologies:

Example

[1] Of course, we already know that we can do calculus on any finite dimensional real vector space. Calculus on infinite dimensional real vector spaces is the realm of functional analysis, and calculus on non-linear finite dimensional spaces is the study of this course.

[2] To differentiate more then once, you need a Riemannian metric.

[3] It turns out that the topology of a smooth manifold is induced by a metric; regardless, we begin with the more general topological space.

- The trivial topology on X where $\mathcal{T} = \{\varnothing, X\}$ (this is the smallest possible topology in the sense of set inclusion).
- The discrete topology on X where $\mathcal{T} = \mathcal{P}(X)$ (largest topology possible in the sense of set inclusion).
- If (X, d) is a metric space then $\mathcal{T} = \{B(p, \varepsilon) : \varepsilon > 0, p \in X\}$ is a topology.

Definition Let $(X, \mathcal{T})$ be a topological space and let $p \in X$. A **neighborhood** (nbhd) U of p is an open set containing p.

Definition Let (p_k) be a sequence in X. We say the (p_k) **converges** to $p \in X$ iff for any nbhd U of p, there exists some $N \in \mathbb{N}$ such that $k \geq N \implies p_k \in U$.

Exercise Show that the above definition is equivalent to the $\varepsilon - \delta$ definition of convergence in a metric space.

Example Metric spaces are very well behaved but a general topological space is not necessarily. For example

- In the trivial topology, any sequence (p_k) converges to any point q (too few open sets).
- In the discrete topology, the only convergent sequences are those which are eventually constant (too many open sets).

In other words, the nature of the topology is very important.

2.2 Continuity

Let $(X, d_1), (Y, d_2)$ be metric spaces. Recall that a map $f : X \to Y$ is **continuous** iff

$$\forall_{p \in X} \forall_{\varepsilon > 0} \exists_{\delta > 0} f(B(p, \delta)) \subseteq B(f(p), \varepsilon)$$

Unfortunately, general topological spaces have no notion of distance so the above definition makes no sense.

Claim *If X, Y are metric spaces and $f : X \to Y$ then f is continuous if and only if for all open sets U in Y we have that $f^{-1}(U)$ is open in X.*

Proof. Suppose the latter holds and let $p \in X, \varepsilon > 0$. Let $U = B(f(p), \varepsilon)$, then U is open in Y so $f^{-1}(U)$ is open in X. Also, $p \in f^{-1}(U)$ because $f(p) \in U$. It follows that there is some $\delta > 0$ such that $B(p, \delta) \subseteq f^{-1}(U)$ and therefore we get that $f(B(p, \delta)) \subseteq U = B(f(p), \varepsilon)$.

Now suppose the former holds, let $U \subseteq Y$ open and let $p \in f^{-1}(U)$. Then $f(p) \in U$ so there is some $\varepsilon > 0$ such that $B(f(p), \varepsilon) \subseteq U$. By assumption, there is some $\delta > 0$ such that $f(B(p, \delta)) \subseteq B(f(p), \varepsilon) \subseteq U$ so $B(p, \delta) \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is open. $\square$

Notice that this reformulation only uses notions of open sets, thus we extend the definition of continuity to topological spaces using it.

Definition Let X, Y be topological spaces. Let $f : X \to Y$ be a function. We say that f is **continuity** iff for all $U \subseteq Y$ open, $f^{-1}(U)$ is open in X.

Lemma *Let X, Y, Z be topological spaces.*

- *Any constant map $f : X \to Y$ is continuous.*
- *The identity map $Id_X : X \to X$ is continuous.*
- *If $f : X \to Y$ and $g : Y \to Z$ are both continuous then $g \circ f : X \to Z$ is continuous.*

Proof.

- Let $U \subseteq Y$ be open. Suppose that $\{q\} \subseteq U$, then $f^{-1}(U) = X$ is open. Otherwise $\{q\} \not\subseteq U$ and $f^{-1}(U) = \varnothing$ is open.
- This is trivial since $f^{-1}(U) = U$.
- $(g \circ f)^{-1}(U) = \{p \in X : g(f(p)) \in U\} = \{p \in X : f(p) \in g^{-1}(U)\} = f^{-1}(g^{-1}(U))$ is open.

$\square$

Definition Let X, Y be topological spaces. A map $f : X \to Y$ is called a **homeomorphism** if f is a bijection and both f, f^{-1} are continuous. We say that X and Y are **homeomorphic** if there is a homeomorphism between them.

Remark Homeomorphism is an equivalence relation (reflexivity and transitivity follow from earlier lemma, symmetry follows by switching f, f^{-1} and noting that $\left(f^{-1}\right)^{-1} = f$).

Homeomorphic topological spaces are "equivalent" in the sense that they preserve topological properties.

Definition Let X be a topological space and let U be an open subset of X. We can define a topology on U by declaring a subset V of U to be open in U iff it is open in X. This is called the **Subspace Topology**[4].

Definition Let X, Y be topological spaces and let $U \subseteq X$ be open. Let $f : X \to Y$ be a map, we write $f|_U : U \to Y$ for the **restriction** of f to U. That is, $f|_U(p) = f(p)$ for all $p \in U$.

Lemma *Let $f : X \to Y$ with X, Y topological spaces.*

- *If f is continuous then so is $f|_U : U \to Y$.*
- *More generally, f is continuous if and only if every $p \in X$ has an open neighborhood U_p in X such that $f|_{U_p}$ is continuous.*

In other words, continuity is a local property.

Proof. Notice first that $f|_U^{-1}(V) = \{p \in U : f(p) \in V\} = U \cap f^{-1}(V)$ for all $V \subseteq Y$.

- Suppose f is continuous and let $V \subseteq Y$ be open, then $f|_U^{-1}(V) = U \cap f^{-1}(V)$ is open as it is the intersection of two open sets.
- Suppose f is continuous, let $p \in X$, and let U_p be any open neighborhood in X of p. By the above, $f|_{U_p}$ is continuous. Conversely suppose every $p \in X$ has an open neighborhood U_p such that $f|_{U_p}$ is continuous. Let $V \subseteq Y$ be open, then $f^{-}1(V) = \bigcup_{p \in X} f^{-1}(V) \cap U_p$. Rewriting the RHS, we get that $f^{-1}(V) = \bigcup_{p \in X} f|_{U_p}^{-1}(V)$ is a union of open sets (by continuity), hence f is continuous.

$\square$

Example Examples of homeomorphisms:

- Let $X = B(p, \delta) \in \mathbb{R}^n$ and consider the map $f : \mathbb{R}^n \to \mathbb{R}^n : q \mapsto q - p$. The map $F = f|_X$ is a homeomorphism $B(p, \delta) \to B(0, \delta)$.
- Let $g : \mathbb{R}^n \to \mathbb{R}^n : p \to {}^1\!/\!_\delta p$ then $g|_{B(0,\delta)}$ is a homeomorphism to $B(0, 1)$.

Together, these examples show that the size of a ball is not preserved by homeomorphisms.

Even worse, consider the map $H : B(0, 1) \to \mathbb{R}^n : p \mapsto {}^p\!/\!_{1-\|p\|}$. Notice that H is invertible with $H^{-1}(q) = {}^q\!/\!_{1+\|q\|}$. Notice further that both maps are continuous so $\mathbb{R}^n$ is homeomorphic to $B(0, 1)$ and the notion of boundedness is not preserved by homeomorphisms.

Remark We will see later that there can be a continuous bijection between topological spaces that is not a homeomorphism (ie: inverse is not continuous).

2.3 Interior and Closure

Definition Let X be a topological space. A subset $F \subseteq X$ is called **closed** if its complement $F^C = X \setminus F$ is open.

Remark A subset A of X can be open, closed, neither, or both.

Example

[4] Technically the subspace topology doesn't require U to be open but we'll get to that later.

- $\varnothing, X$ are closed.
- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Where the last two of the above follow from DeMorgans Laws.

Lemma *Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for all closed $B \subseteq Y$ we have that $f^{-1}(B)$ is closed in X*

Proof. Notice that $f^{-1}(A^c) = \{p \in X : f(p) \in A^c\} = \left[f^{-1}(A)\right]^c$. Given that, the lemma follows by the definition of continuity by taking complements. □

Let A be a subset of X.

Definition We define the **closure** of A, denoted $\mathrm{cl}(A) = \bar{A} = \bigcap_{\substack{F \supseteq A \\ F \text{ closed}}} F$, to be the intersection of all closed sets containing A.

Clearly $\bar{A}$ is closed and $\bar{A} \supseteq A$. Moreover, $\bar{A}$ is the smallest closed set containing A in the sense of set inclusion.

Definition We define the **interior** of A, denoted $\operatorname{int} A = A^\circ = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$ to be the union of all open sets contained in A.

Clearly A° is open and $A^\circ \subseteq A$. Moreover, A° is the largest open set contained in A in the sense of set inclusion.

Lemma

- *A is closed if and only if $A = \bar{A}$.*
- *A is open if and only if $A = A^\circ$.*

Proof.

- If $A = \bar{A}$ then A is closed. Suppose now that A is closed, then A is a closed set containing A and we have $A \subseteq \bar{A} \subseteq A \implies A = \bar{A}$.
- If $A = A^\circ$ then A is open. Suppose now that A is open, then A is a open set contained in A and we have $A \subseteq A^\circ \subseteq A \implies A = A^\circ$.

□

Proposition *Let $A \subseteq X$, then $(A^\circ)^c = \overline{(A^c)}$ and $\left(\bar{A}\right)^c = (A^c)^\circ$. In other words, the complement of the interior is the closure of the complement and the complement of the closure is the interior of the complement.*

Proof. We have

$$A^\circ = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U \implies (A^\circ)^c = \left(\bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U\right)^c = \bigcap_{\substack{U^c \supseteq A^c \\ U^c \text{ closed}}} U^c = \bigcap_{\substack{F \supseteq A^c \\ F \text{ closed}}} F = \overline{(A^c)}$$

and similarly we get that

$$\left(\bar{A}\right)^c = \left(\bigcap_{\substack{F \supseteq A \\ F \text{ closed}}} F\right)^c = \bigcup_{\substack{F^c \subseteq A^c \\ F^c \text{ open}}} F^c = \bigcup_{\substack{U \subseteq A^c \\ U \text{ open}}} U = (A^c)^\circ$$

□

2.4 Exterior and Boundary

Definition Let A be a subset of X. The **exterior** of A is defined to be the interior of its complement. That is, $\operatorname{ext}(A) = \operatorname{int}(A^c)$.

By the previous proposition, we have $\operatorname{ext}(A) = (A^c)^\circ = \left(\bar{A}\right)^c = X \setminus \operatorname{cl}(A)$.

Remark $\operatorname{int}(A) \subseteq A$ and $\operatorname{ext}(A) = \int(A^c) \subseteq A^c$. Since $A \cap A^c = \varnothing$, it follows that the interior and the exterior of A are disjoint open sets.

Definition The **boundary** of A, denoted by ∂A, is defined by $\partial A \coloneqq X \setminus (\operatorname{int}(A) \cup \operatorname{ext}(A)) = (\operatorname{int}(A) \cup \operatorname{ext}(A))^c = (\operatorname{int}(A))^c \cap (\operatorname{ext}(A))^c$.

Remark

- ∂A is always closed since it is the complement of a union of open sets.
- We can write $X = \operatorname{int}(A) \sqcup \partial A \sqcup \operatorname{ext}(A)$ (here we use $\sqcup$ for disjoint union).

From above, we have that $\partial A = (\operatorname{int} A)^c \cap (\operatorname{ext} A)^c$. Recall that $\operatorname{ext}(A) = \operatorname{int}(A^c)$, so $\operatorname{ext}(A)^c = (\operatorname{int}(A^c))^c = \operatorname{cl}((A^c)^c) = \operatorname{cl}(A)$. Therefore, $\partial A = (\operatorname{int} A)^c \cap \bar{A} = \bar{A} \setminus (\operatorname{int} A)$. Hence we have that $X = (\operatorname{int} A) \sqcup \partial A \sqcup (\operatorname{ext} A) = \bar{A} \sqcup (\operatorname{ext} A)$ and also that $\bar{A} = \operatorname{int} A \cup \partial A$. The picture is:

intextdb.png

Thus a set A contains all of its interior points, it contains none of its exterior points and it contains none, some, or all of its boundary points.

Lemma *Let $A \subseteq X$.*

- *$p \in \operatorname{int} A$ iff p has a nbhd contained in A*
- *$p \in \operatorname{ext} A$ iff p has a nbhd contained in A^c.*
- *$p \in \partial A$ iff every nbhd of p contains both a point in A and a point in A^c.*

Proof. The first is obvious since $p \in \operatorname{int} A$ means there is some open set containing p and contained in A - this set is a nbhd of p in A. The second follows from the first. Finlly, to prove the third, recall that $\partial A = X \setminus (\operatorname{int} A \cup \operatorname{ext} A)$ so $p \in \partial A \iff p \notin \operatorname{int}(A)$ and $p \notin \operatorname{ext}(A)$. Using the two previous parts, we see that this happens iff every nbhd of p contains a point in A^c and a point in A. $\square$

Definition Let $A \subseteq X$ and let $p \in X$ (not necessarily in A). We say that p is a **limit point** of A (also called an **acucmulation point** or a **cluster point** of A) if and only if every neighborhood of p contains a point of A other then p.

Lemma *A is closed if and only if A contains all of its limit points.*

Proof.

$$\begin{aligned} A \text{ closed} &\iff A^c \text{ open} \\ &\iff \forall_{p \in X \setminus A} \exists_{U \subseteq X \setminus A \text{ open}} p \in U \\ &\iff \forall_{p \in X \setminus A} p \text{ is not a limit point of } A \\ &\iff \text{every limit point of } A \text{ is contained in } A \end{aligned}$$

$\square$

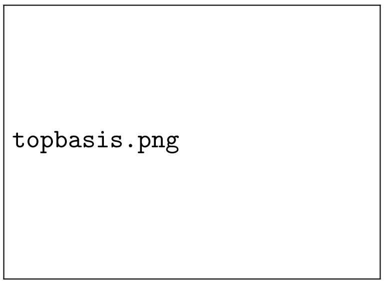
2.5 Basis for Topological Spaces

Let $(X, \mathcal{T})$ be a topological space

Definition A **basis** for the topology $\mathcal{T}$ on X is a sub collection $\mathcal{B} \subseteq \mathcal{T}$ of the open sets with the following property:

- If $U \in \mathcal{T}$ and $p \in U$ then there is some $\mathcal{V} \in \mathcal{B}$ such that $p \in \mathcal{V} \subseteq U$

  Here is a picture:



Remark Any topological space has a basis by simply taking $\mathcal{B} = \mathcal{T}$ (thus taking $\mathcal{V} = U$).

Claim *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$ then any $U \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.*

Proof. Let $U \in \mathcal{T}$, then for all $p \in U \exists_{\mathcal{V}_p \in \mathcal{T}} p \in \mathcal{V}_p \subseteq U$. We then have $U = \bigcup_{p \in U} \mathcal{V}_p$. □

To emphasize: Any open set is a union of basis open sets (no notion of uniqueness unlike a basis in linear algebra). The importance of this is that if a topology $\mathcal{T}$ admits a “simple” basis $\mathcal{B}$ then it is “easier” to deal with. Generally speaking, we want $\mathcal{B}$ to be much smaller then $\mathcal{T}$ since a small basis means the space is “not too big”.

The following shows the usefulness of a simple basis.

Claim *Let X, Y be topological spaces and let $\mathcal{B}$ be a basis for the topology on Y. A map $f : X \to Y$ is continuous iff $f^{-1}(V)$ is open in X for all $V \in \mathcal{B}$.*

Proof. Let $U \subseteq Y$ be open, then $U = \bigcup_{\alpha \in A} \mathcal{V}_\alpha$ for some $\mathcal{V}_\alpha \in \mathcal{B}$. We can verify that

$$f^{-1}(U) = f^{-1}\left(\bigcup_{\alpha \in A} \mathcal{V}_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(\mathcal{V}_\alpha)$$

which is a union of open sets in X by assumption, hence it is open. □

Example Consider $\mathbb{R}^n$ with the standard topology, then $\mathbb{R}^n$ has a countable basis. To see this, let $\mathcal{B} = \{B(q, s) : q \in \mathbb{Q}^n, s \in \mathbb{Q}_{>0}\}$. Clearly $\mathcal{B}$ is countable and its being a basis essentially follows from the density of the rationals.

Notice that $\mathbb{R}^n$ has uncountably many open sets, so this is somewhat remarkable.

Lemma *Let $(X, \mathcal{T})$ be a topological space and suppose there exists a countable basis $\mathcal{B}$ of $\mathcal{T}$. Then any open cover of X admits a countable subcover. In other words, if $X = \bigcup_{\alpha \in A} U_\alpha$ with U_α open then there exists some countable $A' \subseteq A$ such that $X = \bigcup_{\alpha \in A'} U_\alpha$.*

Proof. Let $\mathcal{B}'$ be the subcollection of elements of $\mathcal{B}$ consisting of those basis open sets entirely contained in some U_α for $\alpha \in A$. Notice that $\mathcal{B}'$ is still countable. For each $B \in \mathcal{B}'$ there is some $\alpha(B) \in A$ such that $B \subseteq U_{\alpha(B)}$. Let $p \in X$, then there is some U_α such that $p \in U_\alpha$. So there is some $B \in \mathcal{B}$ such that $p \in B \subseteq U_\alpha$. Hence $B \in \mathcal{B}'$ and so $X = \bigcup_{B \in \mathcal{B}'} B$. Let $A' = \{\alpha(B) : B \in \mathcal{B}'\}$ and notice that A' is countable and that $X = \bigcup_{\alpha \in A'} U_\alpha$. □

We say that X is countably compact of Lindelof if it has the above property.

2.6 Hausdorff Spaces

Definition Let X be a topological space. We say that X is **Hausdorff** iff whenever $p, q \in X$ are distinct points there are U, V open in X such that $p \in U, q \in V$ and $U \cap V = \varnothing$.

The picture is:

`hausdorff.png`

In other words, in a Hausdorff space we can separate points with open sets.

Example

- Suppose (X, d) is a metric space, then X is Hausdorff.
- The trivial topology is not Hausdorff since the only open set containing p and q is X.
- The discrete topology is always Hausdorff.

Lemma *Let X be a Hausdorff topological space, then:*

(a) The singleton sets $\{p\}$ are all closed.

(b) Limits are unique.

Proof.

(a) Let $p \in X$ and pick $q \in X \setminus \{p\}$. Since X is Hausdorff there are U, V open such that $p \in U, q \in V$ and $U \cap V = \varnothing$. This implies that $V \subseteq X \setminus \{p\}$, hence q is an interior point of $X \setminus \{p\}$. Thus since every point of $X \setminus \{p\}$ is an interior point, $X \setminus \{p\}$ is open and $\{p\}$ is closed.

(b) Suppose that (p_k) converges to p and p' where $p \neq p'$. Since X is Hausdorff, there are $U \ni p, V \ni p'$ open with $U \cap V = \varnothing$. However, we eventually have p_k in both U and V, a contradiction.

□

2.7 Topological Manifolds

Definition Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is a **topological manifold of dimension** n (or a **topological** n**-manifold**) if the following three conditions hold:

(i) X is Hausdorff.

(ii) X has a countable basis[5].

(iii) X is locally Euclidean of dimension n.

This is great, of course, but we need to define locally Euclidean.

Definition We say that X is **locally Euclidean of dimension** n if for every $p \in X$ there is some open nbhd U of p and a map $\varphi : U \to \mathbb{R}^n$ such that φ is a homeomorphism from U to $\varphi(U)$. Hence "locally Euclidean of dimension n" means that every point in X has a nbhd that is homeomorphic to an open subset of $\mathbb{R}^n$.

The picture is:

[5] This is a technical condition to ensure the existence of partitions of unity - more later.

`localeuclid.png`

Example

- $X = \mathbb{R}^n$ is a topological n-manifold since $\forall_{p\in X}$ we can take $U = X = \mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ the identity.
- If $U \subseteq \mathbb{R}^n$ is open it is also a topological n-manifold for the same reason.

That is, every open subset of $\mathbb{R}^n$ is (trivially) a topological manifold and we only need one (coordinate) chart.

2.8 Subspace Topology

Let X be a topological space and let $Y \subseteq X$ be a subset of X. Our goal is to define a topology on Y that is induced by the topology on X.

Definition Define $\mathcal{T}_Y \{V \subseteq Y : V = U \cap Y \text{ for some } U \subseteq X \text{ open in } X\}$. This $\mathcal{T}_Y$ is called the **subspace topology**.

We can verify that $\mathcal{T}_Y$ is indeed a topology:

(i) $\varnothing = Y \cap \varnothing$ and $Y = X \cap Y$ and $\varnothing, X$ are open in X.

(ii) If V_α is open in Y for $\alpha \in A$ then there are U_α open in X such that $V_\alpha = U_\alpha \cap Y$. But then we have

$$\bigcup_{\alpha\in A} V_\alpha = \bigcup_{\alpha\in A} U_\alpha \cap Y = Y \cap \left(\bigcup_{\alpha\in A} U_\alpha\right)$$

where $\bigcup_{\alpha\in A} U_\alpha$ is open in X.

(iii) If $V_1, \cdots, V_k$ are open in Y then there are $U_1, \cdot, U_k$ open in U such that $V_i = U_i \cap Y$. Then we have

$$\bigcap_{i=1}^{k} V_i = \bigcap_{i=1}^{k} U_i \cap Y = Y \cap \left(\bigcap_{i=1}^{k} U_i\right)$$

where $\bigcap_{i=1}^{k} U_i$ is open in X.

Remark In the case that $Y \subseteq X$ is open this agrees with our earlier definition since we have

$$V \text{ open in } Y \iff V = U \cap Y \text{ for } U \text{ open in } X \iff V \text{ open in } X \text{ and } V \subseteq Y$$

Example

- Let $\mathbb{R}^k \subseteq \mathbb{R}^n$ by $\mathbb{R}^K = \left\{\left(x^1, \cdot, x^n\right) \in \mathbb{R}^n : x^{k+1} = \cdots = x^n = 0\right\}$, we can verify that the subspace topology on $\mathbb{R}^k$ induced by the metric topology $\mathbb{R}^n$ is exactly the metric topology on $\mathbb{R}^k$.
- Let $X = \mathbb{R}$ and let $Y = [0, 1)$, then $\left[0, \frac{1}{2}\right)$ is open and Y and $\left[\frac{1}{2}, 1\right)$ is closed in Y.

- Let $X = [0,1)$ with the subspace topology induced from $\mathbb{R}$ and let $Y = \mathcal{S}^1$ with the subspace topology of $\mathbb{R}^2$. Define a map $f : X \to Y$ by $f(t) = e^{2\pi i t}$. Clearly f is a continuous bijection but notice that f is not an open map (thus not a homeomorphism) because the image of $\left[0, \frac{1}{2}\right)$ is not open in the topology of Y.

Definition Let X be a topological space and let $Y \subseteq X$ be a topological with the subspace topology. The map $\iota : Y \to X$ taking $\iota(p) = p$ is called the **inclusion map**.

Claim *The subspace topology is the smallest topology that makes the inclusion map continuous.*

Proof. Let $U \subseteq X$ be open and notice that $\iota^{-1}(U) = \{p \in Y : \iota(p) \in U\} = U \cap Y$ must be open in Y. □

Theorem (Characteristic Property of Inclusions) *For any topological space Z, a map $f : Z \to Y$ is continuous iff $\iota \circ f : Z \to X$ is continuous.*

Proof. Suppose first that f is continuous, then since ι is continuous it follows that $\iota \circ f$ is continuous. Suppose now that $\iota \circ f$ is continuous and let $V \subseteq Y$ be open, then $V = Y \cap U$ for some open $U \subseteq X$. We then have

$$(\iota \circ f)^{-1}(U) = \{p \in Z : (\iota \circ f)(p) \in U\} = \{p \in Z : f(p) \in U \cap Y = V\} = f^{-1}(V)$$

is open hence f is continuous. □

We list below several other properties of the subspace topology and inclusion maps

Claim *$f : X \to Z$ continuous $\implies$ $f|_Y : Y \to Z$ continuous.*

Proof. $f|_Y = f \circ \iota : Y \to Z$ but ι is continuous so this composition is as well. □

Claim *If $f : X \to Z$ is continuous then $f : X \to f(X)$ is continuous (here $f(X)$ is a topological space with the subspace topology induced by X).*

Proof.

$$f : X \to f(X) \text{ continuous } \iff \iota \circ f : X \to Z \text{ continuous}$$

□

Claim *A subset $F \subseteq Y$ is closed in Y iff $F = B \cap Y$ for some closed set B in X*

Proof. This essentially follows by taking complements of the corresponding result for open sets. □

Claim *Let $\mathcal{B}$ be a basis for X and let $\mathcal{B}_Y = \{U \cap Y : U \in \mathcal{B}\}$, then $\mathcal{B}_Y$ is a basis for Y.*

Proof. Let $V \subseteq Y$ be open, then there is some U open in X such that $V = U \cap Y$. Let $p \in V$, then $p \in U$ and there is some $B \in \mathcal{B}$ such that $p \in B \subseteq U$. Notice that since $p \in Y$ we also get that $p \in B \cap Y \subseteq U \cap Y = V$ and also that $B \cap Y \in \mathcal{B}_Y$. □

Corollary *If X has a countable basis then so does $Y \subseteq X$.*

Claim *If X is Hausdorff then so is Y*

Proof. Let $p, q \in Y$ with $p \neq q$, then since $p, q \in X$ there are disjoint open sets $U, V \subseteq X$ such that $p \in U$ and $q \in V$. But then $U \cap Y, V \cap Y$ are open in Y (and still disjoint) and we have $p \in U \cap Y$ and $q \in V \cap Y$. □

2.9 Examples of Topological Manifolds

This designates its own section as these are the first topological n-manifolds we've seen that are not just an open subset open subset of $\mathbb{R}^n$ (although we will see many more later).

Example

- Let $U \subseteq \mathbb{R}^n$ be open and let $h : U \to \mathbb{R}^m$ be continuous. Define

$$\Gamma_h :- \left\{(x, y) \in \mathbb{R}^{n+m} : y = h(x)\right\}$$

this is called the **graph** of h. Endow Γ_h with the subspace topology on $\mathbb{R}^{n+m}$ and note that Γ_h is Hausdorff and has a countable basis, thus to show that Γ_h is a topological n-manifold we must only show that it's locally Euclidean of dimension n. Define $\varphi : U \to \Gamma_h$ by $\varphi(x) = (x, h(x))$, then clearly φ is a bijection with inverse $\varphi^{-1}(x, h(x)) = x$ and φ. As a map from $U \to \mathbb{R}^{n+m}$, φ is continuous (this is just calculus) and it follows that φ is continuous from $U \to \Gamma_h$ by subspace topology properties proved earlier. Similarly, φ^{-1} is the restriction to Γ_h of the map $\mathbb{R}^{n+m} \to \mathbb{R}^n : (x, y) \to x$ and is thus also continuous for a similar reason. It follows that φ is a homeomorphism and $\varphi^{-1} : \Gamma_h \to U$ is a **chart** covering all of Γ_h, thus Γ_h is a topological n-manifold.

- The n-spere $\mathcal{S}^n$ is the subset of $\mathbb{R}^{n+1}$ given by $\left\{x \in \mathbb{R}^{n+1} : \|x\|_{\ell_2} = 1\right\}$, we claim that $\mathcal{S}^n$ is a topological n-manifold. As before, it follows immediately that $\mathcal{S}^n$ is both Hausdorff and has a countable basis, so we must only show that it is locally Euclidean. Define for each $1 \leq i \leq n+1$ the sets

$$U_i^+ \coloneqq \left\{x \in \mathcal{S}^n : x^i > 0\right\} \quad \text{and} \quad U_i^- :- \left\{x \in \mathcal{S}^n : x^i < 0\right\}$$

These sets are clearly open and they cover $\mathcal{S}^n$ since

$$x \in \mathcal{S}^n \implies \|x\|_{\ell_2} = 1 \implies x \neq \mathbb{0} \implies \exists_{1 \leq i \leq n+1} x^i \neq 0$$

and since we clearly have $U_i^{\pm} \subseteq \mathcal{S}^n$ for all $1 \leq i \leq n+1$ it follows that the $U_i^{\pm}$'s exactly cover $\mathcal{S}^n$. Since on $U_i^{\pm}$ we have that

$$x^i = \pm\sqrt{1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \left(x^j\right)^2}$$

it follows that the points of $U_i^{\pm}$ lie on the graph of a continuous function in $\mathbb{R}^n$ with domain $\left\{x \in \mathbb{R}^n : \|x\|_{\ell_2} < 1\right\}$ (an open set). We saw in the above example that graphs of continuous functions are homeomorphic to their domain, thus $\mathcal{S}^n$ is covered by open sets that are homeomorphic to $\mathbb{R}^n$ and it follows that $\mathcal{S}^n$ is an n manifold. Here is a picture in $\mathbb{R}^2$ and $\mathbb{R}^3$:

`nspherecover.png`

2.10 Product Topology

Let $X_1, \ldots, X_m$ be topological spaces and let $Y = X_1 \times \cdots \times X_m$. A point $y \in Y$ is an m-tuple $y = (x_1, \ldots, x_m)$ where each $x_i \in X_i$. Our goal will be to endow Y with a natural topology, called the **product topology**.

To do this, we will specify the basis $\mathcal{B} = \{U_1 \times \cdots \times U_m : U_i \subseteq X_i \text{ is open}\}$. To see this is indeed a topology amounts to verifying that $\varnothing, X \in \mathcal{B}$, that arbitrary unions of arbitrary unions of elements of $\mathcal{B}$ are in $\mathcal{B}$, and that finite intersections of arbitrary unions of elements of $\mathcal{B}$ are still unions of elements in $\mathcal{B}$ (all of these are trivial to show).

Example $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the product topology. This is good because it indicates that there is only one natural topology on common sets.

Remark We can take infinite products and still define a topology on the resulting set, and this is the subject of courses in topology, but we will not do that.

Let $X_1, \dots, X_m$ be topological spaces and let Y be the product space $X_1 \times \cdots \times X_m$. Define maps $\pi_i : Y \to X_i$ for all $1 \le i \le m$ by

$$\pi_i (x_1, \dots, x_m) = x_i$$

These π_i's are called the **projection maps onto the i-th** component.

Lemma *The product topology is the smallest topology on Y that ensures that all the projections are continuous.*

Proof. Let $U_i \subseteq X_i$ be open, then $\pi_i^{-1}(U_i) = \{y = (x_1, \dots, x_m) \in Y : \pi_i(y) \in U_i\} = X_1 \times \ldots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_m$. So for π_i to be continuous we need that $\pi_i^{-1}(U_i)$ is open for all i, thus the intersection

$$\pi_1^{-1}(U_1) \cap \cdots \cap \pi_m^{-1}(U_m) = U_1 \times \cdots \times U_m$$

must be open, but this is exactly the product topology. $\square$

Lemma (Characteristic property of the Prodcut Topology) *Let $Y = \prod_{i=1}^m X_i$ be a product space. For any topological space Z, a map $h : Z \to Y$ is continuous iff $h_i := \pi_i \circ h : Z \to X_i$ is continuous.*

Proof. Suppose that h is continuous, then clearly h_i is the composition of continuous functions and therefore continuous. Suppose now that each h_i is continuous and let $W \subseteq Y$ be some basis open set of Y, then $W = U_1 \times \cdots \times U_m$. Notice that

$$\begin{aligned} h^{-1}(W) &= \{q \in Z : h(q) \in W = U_1 \times \cdots \times U_m\} \\ &= \{q \in Z : (\pi_i \circ h)(q) \in U_i \quad \forall_{1 \le i \le m}\} \\ &= h_1^{-1}(U_1) \cap \cdots \cap h_m^{-1}(U_m) \end{aligned}$$

which is a finite intersection of open sets and is therefore open. $\square$

Let $X_1, \dots, X_m$ be topological spaces

Proposition *The product topology is associative in the sense that $(X_1 \times X_2) \times X_3 = X_1 \times (X_2 \times X_3)$ as topological spaces.*

Proof. The set $\{U_1 \times U_2 \times U_3 : U_i \subseteq X_i \text{ is open}\}$ is a basis for both. $\square$

Proposition *For all i and for any $p_j \in X_j$ with $j \neq i$ the map $f_i : X_i \to X_1 \times \cdots \times X_m$ given by $f_i(p) = (p_1, \dots, p_{i-1}, p, p_{i+1}, \dots, p_m)$ is a continuous injection.*

Proof. This map is clearly and injection, and using the characteristic property we see that all of its projections are either constant or the identity, thus continuity follows. $\square$

Proposition *Let $Y_i \subseteq X_i$ be a subspace for all $1 \le i \le m$. The product topology on $Y_1 \times \cdots \times Y_m$ is the same as the subspace topology of $Y_1 \times \cdots \times Y_m \subseteq X_1 \times \cdots \times X_m$.*

Proof. They have the same basis:

$$\left\{ \prod_{i=1}^{m} Y_i \cap S_i : S_i \subseteq X_i \text{ is open} \right\}$$

$\square$

Proposition *If each X_i is Hausdorff then so is $X_1 \times \cdots \times X_m$*

Proof. Let $p = (p_1, \dots, p_m)$ and let $q = (q_1, \dots, q_m)$ be distinct points in $X_1 \times \cdots \times X_m$ and let $1 \le j \le m$ be such that $p_j \neq q_j$. Then there are disjoint open sets $P_j, Q_j \subseteq X_j$ such that $p_j \in P_j$ and $q_j \in Q_j$. It follows that $P := X_1 \times \ldots \times X_{j-1} \times P_j \times X_{j+1} \times X_m$ and $Q := X_1 \times \cdots \times X_{j-1} \times Q_j \times X_{j+1} \times X_m$ are disjoint open sets in Y that separate p and q. $\square$

Proposition *If each X_i has a countable basis then then so does $X_1 \times \cdots \times X_m$*

Proof. Let $\mathcal{B}_i$ be a basis for X_i and let $\mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$. Let $p \in X_1 \times \cdots \times X_m$ and let U be an open set containing p. By definition of the product topology, there are $U_1 \times \cdots \times U_m$ such that $p \in U_1 \times \cdots \times U_m \subseteq U$ and thus $p_i \in U_i$. By definition of $\mathcal{B}_i$, there is some B_i such that $p \in B_i \subseteq U_i$, thus we have

$$p \in B_1 \times \cdots \times B_m \subseteq U_1 \times \cdots \times B_m \subseteq U$$

and since the product of countable sets is countable, it follows that $\mathcal{B}$ is a countable basis for the product space. $\square$

Remark Just as for subspaces, the Hausdorff property and the countable basis property are preserved under this construction.

Definition Let $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ be topological spaces and let $f_i : X_i \to Y_i$ be maps. The **product map** $f_1 \times \cdots \times f_m : X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_m$ is defined by $(f_1 \times \cdots \times f_m)(p_1, \ldots, p_m) = (f_1(p_1), \ldots, f_m(p_m))$.

Lemma *A product of continuous maps is continuous and a product of homeomorphisms is a homeomorphism.*

Proof. Suppose that $f_1, \ldots, f_m$ are continuous and let $V_i \subseteq Y_i$ be open, then we have

$$\begin{aligned}(f_1 \times \cdots \times f_m)^{-1}(V_1 \times \cdots \times V_m) &= \{(p_1, \ldots, p_m) \in X_1 \times \cdots \times X_m : (f_1 \times \cdots \times f_m)(p_1, \ldots, p_m) \in V_1 \times \cdots \times V_m\} \\ &= \{(p_1, \ldots, p_m) \in X_1 \times \cdots \times X_m : f_i(p_i) \in V_i\} \\ &= f_1^{-1}(V_1) \cap \cdots \cap f_m^{-1}(V_m)\end{aligned}$$

which is a finite intersection of continuous sets, thus the product map is continuous.

Suppose now that each f_i is a homeomorphism, then f_i is a bijection with continuous inverse $f_i^{-1} : Y_i \to X_i$. Clearly we have $(f_1 \times \cdots \times f_m)^{-1} = f_1^{-1} \times \cdots \times f_m^{-1}$ which is again the product of continuous functions, thus the product map is a homeomorphism. $\square$

Corollary *Let $M_1, \ldots, M_m$ be topological manifolds of dimension $n_1, \ldots, n_m$ respectively. Then the product $M_1 \times \cdots \times M_m$ is also a topological manifold of dimension $n_1 + \cdots + n_m$.*

Proof. The Hausdorff and countable basis property are immediate so we only show that this is locally Euclidean. Let $p = (p_1, \ldots, p_m) \in M_1 \times \cdots \times M_m$, then $p_i \in M_i$ and there is an open neighborhood $U_i \subseteq M_i$ of p_i and a homeomorphism (chart) $\varphi_i : U_i \to \varphi_i(U_i) \subseteq \mathbb{R}^{n_i}$. By the previous result, we know that $\varphi_1 \times \cdots \times \varphi_m$ is a homeomorphism from the open neighborhood $U_1 \times \cdots \times U_m$ of p in $M_1 \times \cdots \times M_m$ to the open set $\varphi_1(U_1) \times \cdots \times \varphi_m(U_m)$ in $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \cong \mathbb{R}^{n_1 + \cdots + n_m}$. Hence $M_1 \times \cdots \times M_m$ is locally euclidean of dimension $n_1 + \cdots + n_m$. $\square$

Example We can now find even more examples of topological manifolds:

(i) $\mathcal{S}^1 \times \cdots \times \mathcal{S}^1 = T^k$ is a k manifold, the k-torus

(ii) $\mathcal{S}^n \times \mathcal{S}^m$ is an $n + m$ manifold

(iii) $\mathcal{S}^n \times \mathbb{R}^m$ is also an $n + m$ manifold

2.11 The Quotient Topology

Let X be a topological space, let Y be any set, and let $\pi : X \to Y$ be a surjective map.

Definition We endow Y with a topology called the **quotient topology** by declaring a subset $U \subseteq Y$ to be open if and only if $\pi^{-1}(U)$ is open in X.

Claim *This is a topology on Y*

Proof. (i) Clearly $\pi^{-1}(\varnothing) = \varnothing$ and since π is surjective we have that $\pi^{-1}(Y) = X$

(ii) $\pi^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha \pi^{-1}(U_\alpha)$ so arbitrary unions are open.

(iii) $\pi^{-1}\left(\bigcap_{i=1}^{k} U_i\right) = \bigcap_{i=1}^{k} \pi^{-1}(U_i)$

$\square$

Remark The map π is continuous by definition of the topology on Y.

Definition Let X, Y be two topological spaces and let $\pi : X \to Y$ be a continuous surjection. We say that π is a **quotient map** if and only if Y has the quotient topology determined by X and π.

In other words, its true that $U \subseteq Y$ open implies that $\pi^{-1}(U) \subseteq X$ is open since π is continuous, so this says that π is a quotient map if and only if $\pi^{-1}(U) \subseteq X$ open implies $U \subseteq Y$ open.

Why is this called the quotient topology?

The most common way that a quotient topology arises is as follows. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $Y = X/\sim$ be the set of equivalence classes (ie: a point in Y is a subset $[p]$ of X where $[p] = \{q \in X : q \sim p\}$). Let $\pi : X \to Y$ be given by $\pi(p) = [p]$, then π is surjective and we can endow Y with the quotient topology.

Notice that $\pi^{-1}([p]) = \{q \in X : \pi(q) = [p]\} = \{q \in X : q \sim p\} = [p]$. We say that $\pi^{-1}([p]) = \pi^{-1}(\pi(p))$ is called the **fiber** of π over $[p] \in Y$. So X is a disjoint union of fibers.

Let $U \subseteq Y$ be a subset, then

$$\begin{aligned} \pi\left(\pi^{-1}(U)\right) &= \left\{\pi(p) : p \in \pi^{-1}(U)\right\} \\ &= \{\pi(p) : \pi(p) \in U\} = U && \text{because } \pi \text{ is surjective} \end{aligned}$$

Now let $V \subseteq X$ and notice that

$$\pi^{-1}(\pi(V)) = \{p \in X : \pi(p) \in \pi(V)\} \supseteq V$$

but we may not have equality.

Definition A subset $V \subseteq X$ is called **saturated** with respect to π if $V = \pi^{-1}(U)$ for some subset $U \subseteq Y$.

Suppose that V is saturated and observe that $\pi(V) = \pi\left(\pi^{-1}(U)\right) = U$ by surjectivity (so $\pi(V) = U$). Then $\pi^{-1}(\pi(V)) = \pi^{-1}(U) = V$. Hence we conclude that $V \subseteq X$ is saturated if and only if $\pi^{-1}(\pi(V)) = V$.

In other words, a subset V of X is saturated with respect to π if and only if it is a union of fibers of π.

Lemma *A continuous surjection $\pi : X \to Y$ is a quotient map iff it takes saturated open sets to open sets iff it takes saturated closed sets to closed sets.*

Proof. Let V be a saturated open set in X. Then $V = \pi^{-1}(U)$ for some $U \subseteq Y$ and $\pi(V) = \pi\left(\pi^{-1}(U)\right) = U$ hence if $\pi^{-1}(U)$ is open then $\pi(V) = U$ is open as well, but this is what it means for π to be a quotient map

Conversely, if π is a quotient map then $\pi^{-1}(U)$ open in X implies that U is open in Y. But $\pi\left(\pi^{-1}(U)\right)$ is the image of a saturated set.

The statement regarding closed sets is equivalent to the statement about open sets since

$$X \setminus \pi^{-1}(U) = \pi^{-1}(Y \setminus U)$$

so complements of saturated sets are saturated. Combining this with the fact that open and closed are complements, we see these are equivalent. $\square$

Definition A map $f : X \to Y$ is called **open** if $f(W)$ is open in Y whenever W is open in X. It's called closed if $f(K)$ is closed in Y whenever K is closed in X.

Corollary *Let π be a continuous surjection. If π is an open map or a closed map then π is a quotient map.*

Remark Being open or closed is a stronger property then being a quotient map. In other words, a quotient map need not be open or closed.

Example

- Let $X = [0,1)$ and $Y = \mathcal{S}^1$ and $f : [0,1) \to \mathcal{S}^1 : t \mapsto e^{2\pi i t}$. Then f is a continuous surjection and the fibers of f are single points (since f is a bijection). $V = [0, 1/2)$ is saturated and open but $f(V)$ is not open in $\mathcal{S}^1$, hence f is not a quotient map.
- Let $X = [0,1]$ and $Y = \mathcal{S}^1$ and $f : X \to Y : t \mapsto e^{2\pi i t}$. This is again a continuous surjection but this time is not injective. $f^{-1}((1,0)) = \{0,1\}$ hence the fiber of f over $(1,0)$ consists of 2 points $\{0,1\} \subseteq X$ and all other fibers consist of one point.

  Claim *f is a closed map.*

  Proof. Any continuous map between compact Hausdorff topological spaces is a closed map (proof soon). □

  It follows by the corollary that this map is a quotient map.

Lemma *Let $\pi : X \to Y$ and $\rho : Y \to X$ be quotient maps. Then $\rho \circ \pi : X \to Z$ is a quotient map.*

Proof. π, ρ are both continuous surjections, hence $\rho \circ \pi$ is a continuous surjection. Also, U is open in Z iff $\rho^{-1}(U)$ is open in Y iff $\pi^{-1}\left(\rho^{-1}(U)\right)$ is open in X (since they are both quotient maps). But this is just $(\rho \circ \pi)^{-1}(U)$, hence the composition is a quotient map. □